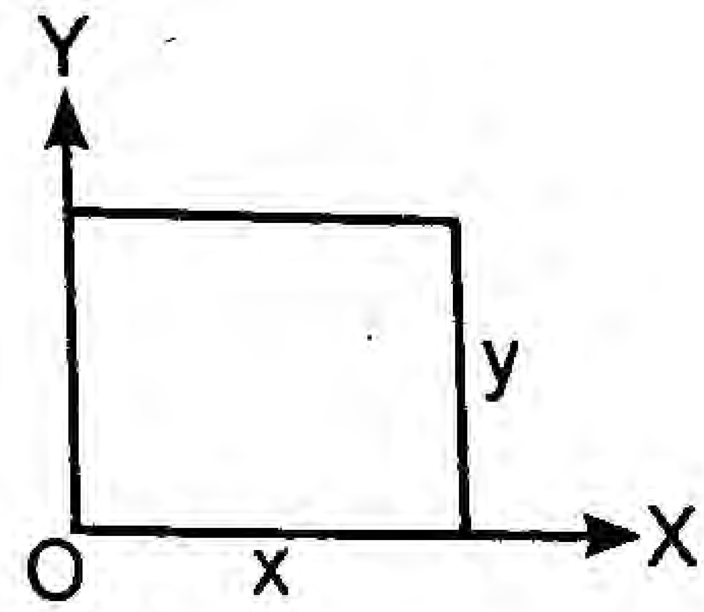
Module–IV: Complex Variable-Differentiation

11 CHAPTER Functions of Complex Variable, Analytic Functions

11.1 INTRODUCTION

The theory of functions of a complex variable is of atmost importance in solving a large number of problems in the field of engineering and science. Many complicated integrals of real functions are solved with the help of functions of a complex variable.

11.2 COMPLEX VARIABLE

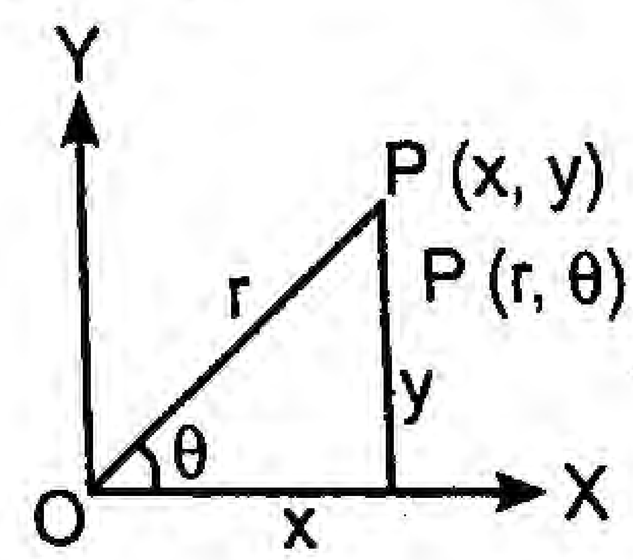


$x + iy$ is a complex variable and it is denoted by z.

(1) $z = x + iy$ where $i = \sqrt{-1}$ (Cartesian form)

(2) $z = r(\cos\theta + i\sin\theta)$ (Polar form)

(3) $z = re^{i\theta}$ (Exponential form)



11.3 FUNCTIONS OF A COMPLEX VARIABLE

$f(z)$ is a function of a complex variable z and is denoted by w.

$$w = f(z)$$
$$w = u + iv$$

where u and v are the real and imaginary parts of $f(z)$.

11.4 LIMIT OF A FUNCTION OF A COMPLEX VARIABLE

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_0. Then the limit of $f(z)$ as z approaches z_0 is w_0.

$$\lim_{z \to z_0} f(z) = w_0$$

11.5 CONTINUITY

The function $f(z)$ of a complex variable z is said to be continuous at the point z_0 if for any given positive number ϵ, we can find a number δ such that $|f(z) - f(z_0)| < \epsilon$ for all points z of the domain satisfying

$$|z - z_0| < \delta$$

$$\Rightarrow \quad \frac{\delta \bar{z}}{\delta z} = \cos 2\theta - i \sin 2\theta$$

Since $\frac{\delta \bar{z}}{\delta z}$ depends on θ. It means for different values of θ, $\frac{\delta \bar{z}}{\delta z}$ has different values.

It means $\frac{\delta \bar{z}}{\delta z}$ has different values for different z. $\quad z = r(\cos\theta + i \sin\theta)$

Therefore $\lim_{\delta z \to 0} \frac{\delta \bar{z}}{\delta z}$ does not tend to a unique limit when $z \neq 0$.

Thus, from (1), it follows that $f'(z)$ is not unique and hence $f(z)$ is not differentiable when $z \neq 0$.

But when $z = 0$ then $f'(z) = 0$ *i.e.*, $f'(0) = 0$ and is unique.

Hence, the function is differentiable at $z = 0$. **Proved.**

By a different method, the above example 1 is again solved as example 2 on page 322.

11.7 ANALYTIC FUNCTION

A function $f(z)$ is said to be **analytic** at a point z_0, if f is differentiable not only at z_0 but at every point of some neighbourhood of z_0.

A function $f(z)$ is analytic in a domain if it is **analytic** at every point of the domain.

The point at which the function is not differentiable is called a **singular point** of the function.

An analytic function is also known as "holomorphic", "regular", "monogenic".

Entire Function. A function which is analytic everywhere (for all z in the complex plane) is known as an entire function.

For Example 1. Polynomials rational functions are entire.

2. $|\bar{z}|^2$ is differentiable only at $z = 0$. So it is no where analytic.

Note: (*i*) An entire is always analytic, differentiable and continuous function. But converse is not true.

(*ii*) Analytic function is always differentiable and continuous. But converse is not true.

(*iii*) A differentiable function is always continuous. But converse is not true.

11.8 THE NECESSARY CONDITION FOR $f(z)$ TO BE ANALYTIC

Theorem. *The necessary conditions for a function $f(z) = u + iv$ to be analytic at all the points in a region R are*

(*i*) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\quad$ (*ii*) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ provided $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist.

Proof: Let $f(z)$ be an analytic function in a region R,

$$f(z) = u + iv,$$

where u and v are the functions of x and y.

Let δu and δv be the increments of u and v respectively corresponding to increments δx and δy of x and y.

$$\therefore \quad f(z + \delta z) = (u + \delta u) + i(v + \delta v)$$

$$\text{Now} \quad \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z} = \frac{\delta u + i\delta v}{\delta z} = \frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}$$

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \text{ or } f'(z) = \lim_{\delta z \to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \quad \ldots (1)$$

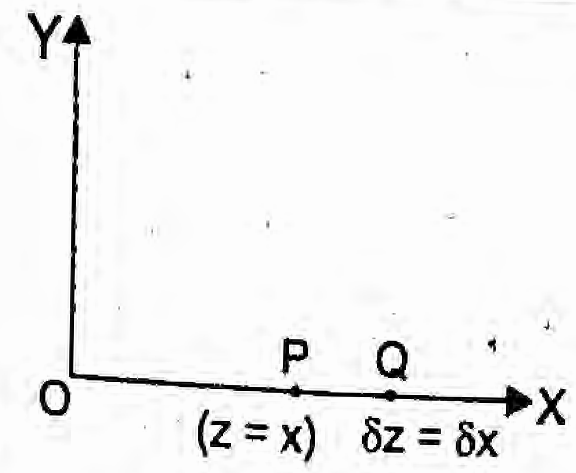
since δz can approach zero along any path.

(*a*) **Along real axis (x-axis)**

$$z = x + iy$$

$$\therefore \quad z = x, \quad \text{but on } x\text{-axis}, y = 0$$

$$\delta z = \delta x, \ \delta y = 0$$



Putting these values in (1), we have

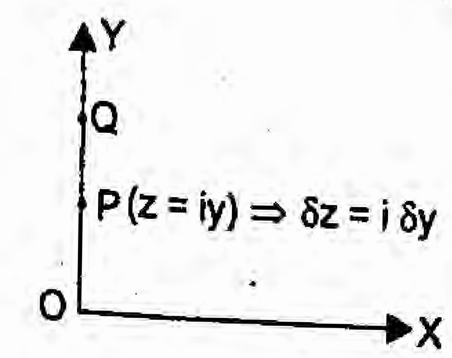
$$f'(z) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \ldots (2)$$

(*b*) **Along imaginary axis (y-axis)**

$$z = x + iy$$

$$z = 0 + iy \quad \text{but on } y\text{-axis}, x = 0$$

$$\delta x = 0, \ \delta z = i\delta y.$$



Putting these values in (1), we get

$$f'(z) = \lim_{\delta y \to 0}\left(\frac{\delta u}{i\delta y} + \frac{i\delta v}{i\delta y}\right) = \lim_{\delta y \to 0}\left(-i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \ldots (3)$$

If $f(z)$ is differentiable, then two values of $f'(z)$ must be the same.

Equating (2) and (3), we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$$

$$\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

are known as Cauchy Riemann equations.

11.9 SUFFICIENT CONDITION FOR $f(z)$ TO BE ANALYTIC

Theorem. *The sufficient condition for a function $f(z) = u + iv$ to be analytic at all the points in a region R are*

(*i*) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(*ii*) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *are continuous functions of x and y in region R.*

Proof. (*i*) Let $f(z)$ be a single-valued function having

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

at each point in the region R. Then the $C - R$ equations are satisfied.

By Taylor's Theorem:

$$f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)$$

$$= u(x, y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + \ldots + i\left[v(x, y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + \ldots\right]$$

$$= [u(x, y) + iv(x, y)] + \left[\frac{\partial u}{\partial x}\cdot\delta x + i\frac{\partial v}{\partial x}\cdot\delta x\right] + \left[\frac{\partial u}{\partial y}\delta y + i\frac{\partial v}{\partial y}\cdot\delta y\right] + \ldots$$

$$= f(z)+\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\delta y+...$$

(Ignoring the terms of second power and higher powers)

$$\Rightarrow \quad f(z+\delta z)-f(z)=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\delta y \quad ...(1)$$

We know $C-R$ equations *i.e.*,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Replacing $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $-\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x}$ respectively in (1), we get

$$f(z+\delta z)-f(z)=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\delta y \quad \text{(taking } i \text{ common)}$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(i\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\right)i\delta y=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\delta x+i\delta y)=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta z$$

$$\Rightarrow \quad \frac{f(z+\delta z)-f(z)}{\delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \boxed{f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}}$$

$$\Rightarrow \quad \boxed{f'(z)=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}}$$

Proved.

Remember: 1. If a function is analytic in a domain D, then u, v satisfy $C-R$ conditions at all points in D.

2. $C-R$ conditions are necessary but not sufficient for analytic function.

3. $C-R$ conditions are sufficient if the partial derivatives are continuous.

Example 3. *Show that the complex variable function $f(z) = |z|^2$ is differentiable only at the origin.*

Solution. $f(z) = |z|^2$ where $z = x + iy$ or $f(z) = x^2 + y^2$

But $\quad f(z) = u + iv \quad \therefore u = x^2 + y^2, v = 0$

$$\frac{\partial u}{\partial x}=2x, \quad \frac{\partial v}{\partial y}=0, \quad \frac{\partial u}{\partial y}=2y, \quad \frac{\partial u}{\partial y}=0$$

If $f(z)$ is differentiable then

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{or} \quad 2x = 0 \text{ or } x = 0$$

and

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \quad \text{or} \quad 2y = 0 \text{ or } y = 0$$

$C-R$ equations are satisfied only when $x = 0, y = 0$.

Thus the given function $f(z)$ is differentiable only at origin. **Proved.**

Example 4. Determine whether $\frac{1}{z}$ is analytic or not?

Solution. Let $w = f(z) = u + iv = \frac{1}{z}$

$$\Rightarrow \quad u + iv = \frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}$$

Equating real and imaginary parts, we get

$$u = \frac{x}{x^2+y^2}, \quad v=\frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}, \quad \frac{\partial u}{\partial y}=\frac{-2xy}{(x^2+y^2)^2}.$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}, \quad \frac{\partial v}{\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

Thus, $\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

Thus $C-R$ equations are satisfied. Also partial derivatives are continuous except at (0, 0).

Therefore $\frac{1}{z}$ is analytic everywhere except at $z = 0$.

Also $\quad \frac{dw}{dz} = -\frac{1}{z^2}$

This again shows that $\frac{dw}{dz}$ exists everywhere except at $z = 0$. Hence $\frac{1}{z}$ is analytic everywhere except at $z = 0$. **Ans.**

Example 5. *Show that the function $e^x (\cos y + i \sin y)$ is an analytic function, find its derivative.*

Solution. Let $e^x (\cos y + i \sin y) = u + iv$

So, $e^x \cos y = u$ and $e^x \sin y = v$ then $\frac{\partial u}{\partial x}=e^x\cos y, \quad \frac{\partial v}{\partial y}=e^x\cos y$

$$\frac{\partial u}{\partial y}=-e^x\sin y, \quad \frac{\partial v}{\partial x}=e^x\sin y$$

Here we see that $\quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

These are $C-R$ equations and are satisfied and the partial derivatives are continuous. Hence, $e^x (\cos y + i \sin y)$ is analytic.

$$f(z) = u + iv = e^x(\cos y + y \sin y) \text{ and } \frac{\partial u}{\partial x}=e^x\cos y, \quad \frac{\partial v}{\partial x}=e^x\sin y$$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=e^x\cos y+ie^x\sin y=e^x(\cos y+i\sin y)=e^x.e^{iy}=e^{x+iy}=e^z.$$

Which is the required derivative. **Ans.**

Example 6. *Using the Cauchy-Riemann equations, show that $f(z) = z^3$ is analytic in the entire z-plane.*

Solution. $f(z)=z^3=(x+iy)^3=x^3+3x^2(iy)+3x(iy)^2+(iy)^3$

$$= x^3 - 3xy^2 + i(3x^2y - y^3)$$

Also $f(z) = u + iv$, $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad \text{... (1)} \qquad \frac{\partial u}{\partial y} = -6xy \quad \text{... (2)}$$

$$\frac{\partial v}{\partial x} = 6xy \quad \text{.... (3)} \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \quad \text{... (4)}$$

From (1) and (4), $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ... (5)

From (2) and (3), $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ... (6)

Thus $C-R$ equations are satisfied and partial derivatives are continuous.
Hence, $f(z)$ is an analytic function. **Proved.**

Example 7. *Test the analyticity of the function $w = \sin z$ and hence derive that:*

$$\frac{d}{dz}(\sin z) = \cos z$$

Solution. $w = \sin z = \sin(x + iy)$
$= \sin x \cos iy + \cos x \sin iy$
$= \sin x \cosh y + i \cos x \sinh y$

$u = \sin x \cosh y$, $v = \cos x \sinh y$

$$\frac{\partial u}{\partial x} = \cos x \cosh y, \frac{\partial u}{\partial y} = \sin x \sinh y \quad \left[\begin{array}{l}\cos iy = \cosh y \\ \sin iy = i \sinh y\end{array}\right]$$

$$\frac{\partial v}{\partial x} = -\sin x \sinh y, \frac{\partial v}{\partial y} = \cos x \cosh y$$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

So $C-R$ equations are satisfied and partial derivatives are continuous.

Hence, $\sin z$ is an analytic function.

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{... (1)}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{... (2)}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{... (3)}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{... (4)}$$

From (1) $\cosh ix = \frac{e^{ix} + e^{-ix}}{2} = \cos x$

From (3) $\cos ix = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$

From (4) $\sin ix = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = i\,\frac{e^x - e^{-x}}{2} = i \sinh x$

From (2) $\sinh ix = \frac{e^{ix} - e^{-ix}}{2} = i \sin x$

$$\frac{d}{dz}(\sin z) = \frac{d}{dz}[\sin x \cosh y + i \cos x \sinh y]$$

$$= \frac{\partial}{\partial x}(\sin x \cosh y + i \cos x \sinh y)$$

$$= \cos x \cosh y - i \sin x \sinh y = \cos x \cos iy - \sin x \sin iy$$

$$= \cos(x + iy) = \cos z \quad \textbf{Ans.}$$

Example 8. *Show that the real and imaginary parts of the function $w = \log z$ satisfy the Cauchy-Riemann equations when z is not zero. Find its derivative.*

Solution. To separate the real and imaginary parts of $\log z$, we put $x = r\cos\theta$; $y = r\sin\theta$

$w = \log z = \log_e(x + iy)$

$\Rightarrow \quad u + iv = \log_e(r\cos\theta + ir\sin\theta) = \log_e r(\cos\theta + i\sin\theta) = \log_e re^{i\theta}$

$$= \log_e r + \log_e e^{i\theta} = \log_e r + i\theta = \log_e \sqrt{x^2 + y^2} + i\tan^{-1}\frac{y}{x} \quad \left[\begin{array}{l} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x}\end{array}\right]$$

So $u = \log_e \sqrt{x^2 + y^2} = \frac{1}{2}\log_e(x^2 + y^2)$, $v = \tan^{-1}\frac{y}{x}$

On differentiating u, v, we get

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2 + y^2}.(2x) = \frac{x}{x^2 + y^2} \quad \text{... (1)}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2} \quad \text{... (2)}$$

From (1) and (2), $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ... (A)

Again differentiating u, v, we have

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2 + y^2}(2y) = \frac{y}{x^2 + y^2} \quad \text{... (3)}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2} \quad \text{... (4)}$$

From (3) and (4), we have

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{... (B)}$$

Equations (A) and (B) are $C-R$ equations and partial derivatives are continuous.
Hence, $w = \log z$ is an analytic function except
when $x^2 + y^2 = 0 \Rightarrow x = y = 0 \Rightarrow x + iy = 0 \Rightarrow z = 0$
Now $w = u + iv$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2}$$

$$= \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{x + iy} = \frac{1}{z}$$

Which is the required derivative. **Ans.**

Example 9. *Find the point where the Cauchy-Riemann equations are satisfied for the function:*

$f(z) = xy^2 + ix^2y$. Where does $f'(z)$ exist? Where $f(z)$ is analytic?

Solution. We have, $f(z) = xy^2 + ix^2y$, $\quad f(z) = u + iv$

$u = xy^2$, $\quad v = x^2y$

$$\frac{\partial u}{\partial x} = y^2, \qquad \frac{\partial v}{\partial x} = 2xy$$

$$\frac{\partial u}{\partial y} = 2xy, \qquad \frac{\partial v}{\partial y} = x^2$$

If $f(z)$ is an analytic function, then it will satisfy $C-R$ equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{i.e. } y^2 = x^2 \quad \text{... (1)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ i.e. } 2xy = -2xy \text{ or } 4xy = 0 \quad \dots(2)$$

Solving (1) and (2), we get $x = y = 0$

At origin $C-R$ equations are satisfied, $f'(z)$ exists at origin only and no where else. Hence $f(z)$ is analytic at origin only. **Ans.**

Example 10. *Find the values of C_1 and C_2 such that the function $f(z) = x^2 + C_1 y^2 - 2xy + i(C_2x^2 - y^2 + 2xy)$ is analytic. Also find $f'(z)$.* **(AKTU, 2016-2017)**

Solution. Let $f(z) = u + iv = x^2 + C_1y^2 - 2xy + i(C_2x^2 - y^2 + 2xy)$

Equating real and imaginary parts, we get

$$u = x^2 + C_1y^2 - 2xy \text{ and } v = C_2x^2 - y^2 + 2xy$$

$$\frac{\partial u}{\partial x} = 2x - 2y \text{ and } \frac{\partial v}{\partial x} = 2C_2x + 2y$$

$$\frac{\partial u}{\partial y} = 2C_1y - 2x \text{ and } \frac{\partial v}{\partial y} = -2y + 2x$$

$C-R$ equations are

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}\right] \Rightarrow \begin{aligned}2x - 2y &= -2y + 2x \quad \dots(1)\\ 2C_1y - 2x &= -2C_2x - 2y \quad \dots(2)\end{aligned}$$

From (2) equating the coefficient of x and y.

$$2C_1 = -2 \Rightarrow C_1 = -1$$
$$-2 = -2C_2 \Rightarrow C_2 = 1$$

Hence, $C_1 = -1$ and $C_2 = 1$ **Ans.**

On putting the value of C_2, we get

$$\frac{\partial u}{\partial x} = 2x - 2y, \qquad \frac{\partial v}{\partial x} = 2x + 2y$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (2x-2y) + i(2x+2y) = 2[(x+ix) + (-y+iy)]$$
$$= 2[(1+i)x + i(1+i)y]$$
$$= 2(1+i)(x+iy) = 2(1+i)z$$

This is the required derivative. **Ans.**

Example 11. *Show that the function $z\,|z|$ is not analytic anywhere.*

Solution. Let $w = z\,|z|$

$$w = u + iv \text{ and } z = x + iy \qquad |z| = \sqrt{x^2+y^2}$$

$$w = z\,|z| \Rightarrow u + iv = (x+iy)\sqrt{x^2+y^2}$$

$$u = x\sqrt{x^2+y^2} \text{ and } v = y\sqrt{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \sqrt{x^2+y^2} + \frac{x\cdot 2x}{2\sqrt{x^2+y^2}}, \qquad \frac{\partial v}{\partial y} = \sqrt{x^2+y^2} + \frac{y\cdot 2y}{2\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial x} = \frac{x^2+y^2+x^2}{\sqrt{x^2+y^2}} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}} \quad \dots(1) \qquad = \frac{x^2+y^2+y^2}{\sqrt{x^2+y^2}} = \frac{x^2+2y^2}{\sqrt{x^2+y^2}} \quad \dots(2)$$

Also

$$\frac{\partial u}{\partial y} = \frac{x\cdot 2y}{2\sqrt{x^2+y^2}} \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{y\cdot 2x}{2\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{xy}{\sqrt{x^2+y^2}} \quad \dots(3) \qquad \frac{\partial v}{\partial x} = \frac{xy}{\sqrt{x^2+y^2}} \quad \dots(4)$$

Case I. When $x \neq y$,

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \text{[From (1) and (2)]}$$

$$\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x} \quad \text{[From (3) and (4)]}$$

Both $C-R$ equations are not satisfied

Thus, $z\,|z|$ is not analytic when $x \neq y$. ... (5)

Case II. When $x = y$

From (1), $\frac{\partial u}{\partial x} = \frac{2y^2+y^2}{\sqrt{y^2+y^2}} = \frac{3y^2}{\sqrt{2y^2}} = \frac{3y}{\sqrt{2}}$ | From (2), $\frac{\partial v}{\partial y} = \frac{y^2+2y^2}{\sqrt{y^2+y^2}} = \frac{3y^2}{\sqrt{2y^2}} = \frac{3y}{\sqrt{2}}$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

From (3), $\frac{\partial u}{\partial y} = \frac{y^2}{\sqrt{y^2+y^2}} = \frac{y}{\sqrt{2}}$ | From (4), $\frac{\partial v}{\partial x} = \frac{y^2}{\sqrt{y^2+y^2}} = \frac{y}{\sqrt{2}}$

$$\Rightarrow \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

Here first $C-R$ equation is satisfied but not second.

Thus $z\,|z|$ is not analytic when $x = y$. ... (6)

From (5) and (6) we conclude that $z\,|z|$ is not analytic anywhere. **Proved.**

Example 12. *Discuss the analyticity of the function $f(z) = z\bar{z}$.*

Solution. $f(z) = z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2 + y^2$

$$f(z) = x^2 + y^2 = u + iv.$$
$$u = x^2 + y^2,\ v = 0$$

At origin,

$$\frac{\partial u}{\partial x} = \lim_{h\to 0}\frac{u(0+h,\ 0) - u(0,0)}{h} = \lim_{h\to 0}\frac{h^2}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0}\frac{u(0,0+k) - u(0,0)}{k} = \lim_{k\to 0}\frac{k^2}{k} = 0$$

Also,

$$\frac{\partial v}{\partial x} = \lim_{h\to 0}\frac{v(0+h,0) - v(0,0)}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0}\frac{v(0,0+k) - v(0,0)}{k} = 0$$

Thus, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Hence, $C-R$ equations are satisfied at the origin.

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{(x^2+y^2)-0}{x+iy}$$

Let $z \to 0$ along the line $y = mx$, then

$$f'(0) = \lim_{x\to 0} \frac{(x^2+m^2x^2)}{(x+imx)} = \lim_{x\to 0} \frac{(1+m^2)x}{1+im} = 0$$

Therefore, $f'(0)$ is unique. Hence the function $f(z)$ is analytic at $z = 0$ **Ans.**

Example 13. *Show that the function $f(z) = u + iv$, where*

$$f(z) = \begin{cases} \dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

satisfied the Cauchy-Riemann equations at $z = 0$. Is the function analytic at $z = 0$? Justify your answer. (MDU Dec 2009)

Solution. $$f(z) = \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} = u+iv$$

$$u = \frac{x^3-y^3}{x^2+y^2}, \qquad v = \frac{x^3+y^3}{x^2+y^2}$$

[By differentiation the value of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at (0, 0) we get $\frac{0}{0}$, so we apply first principle method]

At the origin

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(0+h,\ 0)-u(0,\ 0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2}}{h} = 1 \qquad \text{(Along } x\text{- axis)}$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(0,\ 0+k)-u(0,\ 0)}{k} = \lim_{k\to 0} \frac{\frac{-k^3}{k^2}}{k} = -1 \qquad \text{(Along } y\text{- axis)}$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(0+h,\ 0)-v(0,\ 0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2}}{h} = 1 \qquad \text{(Along } x\text{- axis)}$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0} \frac{v(0,\ 0+k)-v(0,\ 0)}{k} = \lim_{k\to 0} \frac{\frac{k^3}{k^2}}{k} = 1 \qquad \text{(Along } y\text{-axis)}$$

Thus we see that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, Cauchy-Riemann equations are satisfied at $z = 0$.

Again $$f'(0) = \lim_{z\to 0} \frac{f(0+z)-f(0)}{z} = \lim_{z\to 0} \left[\frac{\frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2} - (0)}{x+iy} \right]$$

$$= \lim_{z\to 0} \left[\frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2} \cdot \frac{1}{x+iy} \right] \qquad \text{[Increment} = z]$$

Now let $z \to 0$ along $y = x$, then

$$f'(0) = \lim_{x\to 0} \frac{x^3-x^3+i(x^3+x^3)}{x^2+x^2} \left(\frac{1}{x+ix} \right)$$

$$= \frac{2i}{2(1+i)} = \frac{i}{1+i} = \frac{i(1-i)}{(1+i)(1-i)} = \frac{i+1}{1+1} = \frac{1}{2}(1+i) \qquad \text{... (1)}$$

Again let $z \to 0$ along $y = 0$, then

$$f'(0) = \lim_{x\to 0} \frac{x^3+ix^3}{x^2} \cdot \frac{1}{x} = (1+i) \qquad \text{... (2)}$$

From (1) and (2), we see that $f'(0)$ is not unique. Hence the function $f(z)$ is not analytic at $z = 0$. **Ans.**

Example 14. *Show that the function defined by $f(z) = \sqrt{|xy|}$ Satisfied Cauchy-Riemann equation at the origin but is not analytic at that point.*

Solution. Let $f(z) = u + iv = \sqrt{|xy|}$

Equating real and imaginary parts, we get $u = \sqrt{|xy|}$, $v = 0$

At origin

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(0+h,\ 0)-u(0,\ 0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(0,\ 0+k)-u(0,\ 0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

Also

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(0+h,\ 0)-v(0,\ 0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0} \frac{v(0,\ 0+k)-v(0,\ 0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

From the above results, it is clear that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, C– R equations are satisfied at the origin

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{\sqrt{|xy|}-0}{x+iy}$$

Let $z \to 0$ along the line $y = mx$, then

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|mx^2|}-0}{x(1+im)} = \lim_{x\to 0} \frac{\sqrt{|m|}}{1+im}$$

Thus, the limit on R.H.S. depends upon m and hence will have different values for different values of m.

Therefore, $f'(0)$ is not unique.

Hence the function $f(z)$ is not analytic at $z = 0$. **Proved.**

Example 15. *Show that the function*

$$f(z) = e^{-z^{-4}}, \quad (z \neq 0) \quad \text{and}$$
$$f(0) = 0$$

is not analytic at z = 0, although, Cauchy-Riemann equations are satisfied at the point. How would you explain this.

Solution.

$$f(z) = u + iv = e^{-z^{-4}} = e^{-(x+iy)^{-4}} = e^{-\frac{1}{(x+iy)^4}}$$

$$\Rightarrow \quad u + iv = e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}} = e^{-\frac{1}{(x^2+y^2)^4}[(x^4+y^4-6x^2y^2)-i4xy(x^2-y^2)]}$$

$$\Rightarrow \quad u+iv = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} . e^{\frac{-i4xy(x^2-y^2)}{(x^2+y^2)^4}}$$

$$\Rightarrow \quad u+iv = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}}\left[\cos\frac{4xy(x^2-y^2)}{(x^2+y^2)^4} - i\sin\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right]$$

Equating real and imaginary parts, we get

$$u = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}}\cos\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}, \quad v = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}}\sin\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}$$

At $z = 0$

$$\frac{\partial u}{\partial x} = \lim_{h\to 0}\frac{u(0+h,\ 0)-u(0,\ 0)}{h} = \lim_{h\to 0}\frac{e^{-h^{-4}}}{h} = \lim_{h\to 0}\frac{1}{h e^{\frac{1}{h^4}}}$$

$$= \lim_{h\to 0}\left[\frac{1}{h\left[1+\frac{1}{h^4}+\frac{1}{2!h^8}+\frac{1}{3!h^{12}}+\ldots\right]}\right], \quad \left(e^x = 1+x+\frac{x^2}{2!}+\ldots\right)$$

$$= \lim_{h\to 0}\left[\frac{1}{\left[h+\frac{1}{h^3}+\frac{1}{2h^7}+\frac{1}{6h^{11}}+\ldots\right]}\right] = \frac{1}{0+\infty} = \frac{1}{\infty} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0}\frac{u(0,\ 0+k)-u(0,\ 0)}{k} = \lim_{k\to 0}\frac{e^{-k^{-4}}}{k} = \lim_{k\to 0}\frac{1}{k e^{\frac{1}{k^4}}} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0}\frac{v(0+h,\ 0)-v(0,\ 0)}{h} = \lim_{h\to 0}\frac{e^{-h^{-4}}}{h} = \lim_{h\to 0}\frac{1}{h.e^{\frac{1}{h^4}}} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0}\frac{v(0,\ 0+k)-v(0,\ 0)}{k} = \lim_{k\to 0}\frac{e^{-k^{-4}}}{k} = \lim_{k\to 0}\frac{1}{k.e^{\frac{1}{k^4}}} = 0$$

Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ($C - R$ equations are satisfied at $z = 0$)

But $$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{e^{-z^{-4}}}{z}$$

Along $z = re^{i\frac{\pi}{4}}$

$$f'(0) = \lim_{r\to 0}\frac{e^{-r^{-4}} \cdot e^{-\left(e^{i\frac{\pi}{4}}\right)^{-4}}}{re^{i\frac{\pi}{4}}} = \lim_{r\to 0}\frac{e^{-r^{-4}} \cdot e^{-\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{-4}}}{re^{i\frac{\pi}{4}}}$$

$$= \lim_{r\to 0}\frac{e^{-r^{-4}}\ e^{-\cos\pi}}{re^{i\frac{\pi}{4}}} = \lim_{r\to 0}\frac{e^{-r^{-4}} \cdot e}{re^{i\frac{\pi}{4}}} = \infty$$

Showing that $f'(z)$ does not exist at $z = 0$. Hence $f(z)$ is not analytic at $z = 0$. **Proved.**

Example 16. *Show that the function f (z) defined by* [AKTU 2014-15]

$$f(z) = \begin{cases} \dfrac{x^3y^5(x+iy)}{x^6+y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not anlytic at the origin even though it satisfies Cauchy Riemann equations at the origin. (UP. III Semester 2011)

Solution: Here $f(z) = u + iv = \dfrac{x^3y^5(x+iy)}{x^6+y^{10}}$, $z \neq 0$

Equating real and imaginary parts, we get

$$u = \frac{x^4y^5}{x^6+y^{10}}, \quad v = \frac{x^3y^6}{x^6+y^{10}}$$

$$\frac{\partial u}{\partial x} = \lim_{h\to 0}\frac{u(0+h,0)-u(0,0)}{h} = \lim_{h\to 0}\frac{\frac{0}{h^6}}{h} = \lim_{h\to 0}\frac{0}{h^7} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0}\frac{u(0,0+k)-u(0,0)}{k} = \lim_{k\to 0}\frac{\frac{0}{k^{10}}}{k} = \lim_{k\to 0}\frac{0}{k^{11}} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0}\frac{v(0+h,0)-v(0,0)}{h} = \lim_{h\to 0}\frac{\frac{0}{h^6}}{h} = \lim_{h\to 0}\frac{0}{h^7} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0}\frac{v(0,0+k)-v(0,0)}{k} = \lim_{k\to 0}\frac{\frac{0}{k^{10}}}{k} = \lim_{k\to 0}\frac{0}{k^{11}} = 0$$

From the above results, it is clear that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Hence, $C - R$ equations are satisfied at the origin.

But $$f'(0) = \lim_{z\to 0}\frac{f(0+z)-f(0)}{z} = \lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{x^3y^5(x+iy)}{x^6+y^{10}} - 0\right]\frac{1}{x+iy}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3y^5}{x^6+y^{10}}$$

Let $z \to 0$ along the radius vector $y = mx$, then

$$f'(y) = \lim_{x \to 0} \frac{m^5 x^8}{x^6 + m^{10} x^{10}} = \lim_{x \to 0} \frac{m^5 x^2}{1 + m^{10} x^4} = \frac{0}{1} = 0 \quad \text{...(1)}$$

Again $z \to 0$ along the curve $y^5 = x^3$

$$f'(0) = \lim_{x \to 0} \frac{x^6}{x^6 + x^6} = \frac{1}{2} \quad \text{...(2)}$$

(1) and (2) show that $f'(0)$ does not exist. Hence, $f(z)$ is not analytic at origin although Cauchy Riemann equations are satisfied.

Example 17. *Examine the nature of the function*

$$f(z) = \begin{cases} \dfrac{x^3 y (y - ix)}{x^6 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Prove that $\dfrac{f(z) - f(0)}{z} \to 0$ *as* $z \to 0$ *along any radius vector but not as* $z \to 0$ *in any manner and also that* $f(z)$ *is not analytic at* $z = 0$.

Solution. Here, $\quad f(z) = u + iv = \dfrac{x^3 y (y - ix)}{x^6 + y^2}, z \neq 0$

$$u = \frac{x^3 y^2}{x^6 + y^2}, \quad v = -\frac{x^4 y}{x^6 + y^2}$$

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0+h, 0) - u(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{0}{h^6}}{h} = \lim_{h \to 0} \frac{0}{h^7} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0, 0+k) - u(0, 0)}{k} = \lim_{k \to 0} \frac{-\frac{0}{k^2}}{k} = \lim_{k \to 0} \frac{0}{k^3} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h, 0) - v(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{0}{h^6}}{h} = \lim_{h \to 0} \frac{0}{h^7} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0, 0+k) - v(0, 0)}{k} = \lim_{k \to 0} \frac{-\frac{0}{k^2}}{k} = \lim_{k \to 0} \frac{0}{k^3} = 0$$

From the above results, it is clear that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, C-R equations are satisfied at the origin

$$\frac{f(z) - f(0)}{z} = \left[\frac{x^3 y (y - ix)}{x^6 + y^2} - 0\right] \cdot \frac{1}{x + iy}$$

$$= \frac{-ix^3 y (x + iy)}{(x^6 + y^2)} \cdot \frac{1}{x + iy} = -i \frac{x^3 y}{x^6 + y^2}$$

Let $z \to 0$ along radius vector $y = mx$ then,

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{x \to 0} \frac{-ix^3 (mx)}{x^6 + m^2 x^2} = \lim_{x \to 0} \frac{-imx^2}{x^4 + m^2} = 0$$

Hence, $\dfrac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector.

Now let $z \to 0$ along a curve $y = x^3$ then,

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{x \to 0} \frac{-ix^3 x^3}{x^6 + x^6} = \frac{-i}{2}$$

Hence, $\dfrac{f(z) - f(0)}{z}$ does not tend to zero as $z \to 0$ along the curve $y = x^3$.

We observe that $f'(0)$ does not exist hence $f(z)$ is not analytic at $z = 0$. **Ans.**

11.10 C-R EQUATIONS IN POLAR FORM

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$$

(RGPV., K.U. 2009, Bhopal, III Sem. Dec. 2007)

Proof. We know that $x = r \cos \theta$, and u is a function of x and y.

$$z = x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

$$u + iv = f(z) = f(re^{i\theta}) \quad \text{...(1)}$$

Differentiating (1) partially w.r.t., "r", we get

$$\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} = f'(re^{i\theta}).e^{i\theta} \quad \text{...(2)}$$

Differentiating (1) w.r.t. "θ", we get

$$\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} = f'(re^{i\theta}) re^{i\theta} i \quad \text{...(3)}$$

Substituting the value of $f'(re^{i\theta})e^{i\theta}$ from (2) in (3), we get

$$\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} = r\left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r}\right)i \quad \text{or} \quad \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} = ir \frac{\partial u}{\partial r} - r \frac{\partial v}{\partial r}$$

Equating real and imaginary parts, we get

$$\boxed{\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}} \quad \Rightarrow \quad \frac{\partial v}{\partial r} = \frac{-1}{r} \frac{\partial u}{\partial \theta}$$

And $\quad \boxed{\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}}$ **Proved.**

Example 18. *Find p such that the function $f(z)$ expressed in polar coordinates as $f(z) = r^2 \cos 2\theta + ir^2 \sin p\theta$ is analytic.*

Solution. We know that

$$f(z) = u + iv$$

Here, $u = r^2 \cos 2\theta$ and $v = r^2 \sin p\theta$

$$\frac{\partial u}{\partial r} = 2r \cos 2\theta \quad \text{...(1)}$$

$$\frac{\partial v}{\partial \theta} = pr^2 \cos p\theta \quad \text{...(2)}$$

C-R-equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \ldots(3)$$

On putting the values of $\frac{\partial u}{\partial r}$ and $\frac{\partial v}{\partial \theta}$ from (1) and (2) in (3), we get

$$2r \cos 2\theta = pr \cos p\,\theta$$

$$2 \cos 2\theta = p \cos p\,\theta$$

$$\Rightarrow \qquad p = 2 \qquad \textbf{Ans.}$$

11.10 DERIVATIVE OF w IN POLAR FORM

We know that $w = u + iv$, $\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

But
$$\frac{dw}{dz} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial w}{\partial r}\cos\theta - \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\frac{\sin\theta}{r}$$

$$= \frac{\partial w}{\partial r}\cos\theta - \left(-r\frac{\partial v}{\partial r} + i\cdot r\frac{\partial u}{\partial r}\right)\frac{\sin\theta}{r} \qquad \left[\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r},\ \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}\right]$$

$$= \frac{\partial w}{\partial r}\cos\theta - i\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\sin\theta$$

$$= \frac{\partial w}{\partial r}\cos\theta - i\frac{\partial}{\partial r}(u + iv)\sin\theta = \frac{\partial w}{\partial r}\cos\theta - i\,\frac{\partial w}{\partial r}\sin\theta \qquad [\because w = u + iv]$$

$$= (\cos\theta - i\sin\theta)\,\frac{\partial w}{\partial r}$$

Second form of $\frac{\partial w}{dz}$

$$\frac{dw}{dz} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial (u + iv)}{\partial r}\cos\theta - \frac{\partial w}{\partial \theta}\,\frac{\sin\theta}{r} \qquad [w = u + iv]$$

$$= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cos\theta - \frac{\partial w}{\partial \theta}\,\frac{\sin\theta}{r}$$

$$= \left(\frac{1}{r}\frac{\partial v}{\partial \theta} - i\cdot\frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta - \frac{\partial w}{\partial \theta}\cdot\frac{\sin\theta}{r}$$

$$= -\frac{i}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\cos\theta - \frac{\partial w}{\partial \theta}\left(\frac{\sin\theta}{r}\right)$$

$$= -\frac{i}{r}\frac{\partial}{\partial \theta}(u + iv)\cos\theta - \frac{\partial w}{\partial \theta}\left(\frac{\sin\theta}{r}\right)$$

$$= -\frac{i}{r}\frac{\partial w}{\partial \theta}\cos\theta - \frac{\partial w}{\partial \theta}\,\frac{\sin\theta}{r}$$

$$= -\frac{i}{r}(\cos\theta - i\sin\theta)\,\frac{\partial w}{\partial \theta}$$

$$\boxed{\frac{dw}{dz} = (\cos\theta - i\sin\theta)\,\frac{\partial w}{\partial r}} \qquad \left[-\frac{i}{r}\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial r}\right]$$

$$\boxed{\frac{dw}{dz} = -\frac{i}{r}(\cos\theta - i\sin\theta)\,\frac{\partial w}{\partial \theta}}$$

These are the two forms for $\frac{dw}{dz}$.

Example 10. *If n is real, show that $r^n(\cos n\theta + i\sin n\theta)$ is analytic except possible when $r = 0$ and that its derivative is $nr^{n-1}[\cos(n-1)\theta + i\sin(n-1)\theta]$.*

Solution. Let
$$w = f(z) = u + iv = r^n(\cos n\theta + i\sin n\theta)$$

Here,
$$u = r^n\cos n\theta, \qquad v = r^n\sin n\theta$$

then,
$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta, \qquad \frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$$

$$\frac{\partial u}{\partial \theta} = -nr^n\sin n\theta, \qquad \frac{\partial v}{\partial \theta} = nr^n\cos n\theta$$

Here,
$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta = \frac{1}{r}(nr^n\cos n\theta)$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \ldots(1)$$

and
$$\frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$$

$$\Rightarrow \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}(-nr^n\sin n\theta)$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad \ldots(2)$$

Equations (1) and (2) satisfied C-R equations.

We have,
$$\frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r} = (\cos\theta - i\sin\theta)\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$= (\cos\theta - i\sin\theta)(nr^{n-1}\cos n\theta + inr^{n-1}\sin n\theta)$$

$$= (\cos\theta - i\sin\theta)\,nr^{n-1}(\cos n\theta + i\sin n\theta)$$

$$= nr^{n-1}\{(\cos n\theta\cos\theta + \sin n\theta\sin\theta) + i(\sin n\theta\cos\theta - \cos n\theta\sin\theta)\}$$

$$= nr^{n-1}\{\cos(n-1)\theta + i\sin(n-1)\theta\}$$

This exists for all finite values of r including zero, except when $r = 0$ and $n \le 1$. (except at $z = 0$) **Proved.**

EXERCISE 11.1

Determine which of the following functions are analytic:

1. $x^2 + iy^2$ — **Ans.** Analytic at all points $y = x$
2. $2xy + i(x^2 - y^2)$ — **Ans.** Not analytic
3. $\frac{x - iy}{x^2 + y^2}$ — **Ans.** Not analytic
4. $\frac{1}{(z-1)(z+1)}$ — **Ans.** Analytic at all points, except at $z = \pm 1$
5. $\frac{x - iy}{x - iy + a}$ — **Ans.** Not analytic

Ans. Yes, analytic

6. $\sin x \cosh y + i \cos x \sinh y$ **Ans.** Yes, analytic at origin

7. $xy + iy^2$

8. Discuss the analyticity of the function $f(z) = z\bar{z} + \bar{z}^2$ in the complex plane, where $\bar{z}$ is the complex conjugate of z. Also find the points where it is differentiable but not analytic. **Ans.** Differentiable only at $z = 0$, No where analytic.

9. Show the function of $\bar{z}$ is not analytic any where.

10. For what values of z do the function w defined by the following equation, ceases to be analytic? $w = \sin u \cosh v + i \cos u \sinh v$. **Ans.** $z \neq 1$

11. Show that the function $w = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$ is an analytic function find $\frac{dw}{dz}$. **Ans.** $-\frac{1}{z^2}$

12. Examine the nature of the function

$$f(z) = \frac{x^2 y^5 (x+iy)}{x^4 + y^{10}}; z \neq 0, f(0) = 0$$ *in the region including the origin.* **Ans.** Not analytic

Choose the correct answer :

13. In order that the function $f(z) = \frac{|z|^2}{z}, z \neq 0$ be continuous at $z = 0$, we should define $f(0)$ equal to
(a) 2 (b) – 1 (c) 0 (d) 1 **Ans.** (c)

14. If $f(z)$ is analytic and equal to $u(x, y) + iv(x, y)$ then $f'(z)$ equals.
(a) $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (b) $\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}$ (c) $\frac{\partial u}{\partial y} - i\frac{\partial v}{\partial x}$ (d) none of these **Ans.** (a)

15. The only function among the following, that is analytic, is :
(a) If $f(z) = Re(z)$ (b) $f(z) = Im(z)$ (c) $f(z) = \bar{z}$ (d) $f(z) = \sin z$. **Ans.** (d)

16. The Cauchy-Riemann equations for $f(z) = u(x, y) + iv(x, y)$ to be analytic are :
(a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ (b) $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
(c) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (d) $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ **Ans.** (c)
(R.G.P.V., Bhopal, III Semester, Dec. 2006)

17. Polar form of *C-R* equations are :
(a) $\frac{\partial u}{\partial \theta} = \frac{1}{r}\frac{\partial v}{\partial r}, \ \frac{\partial u}{\partial r} = r\frac{\partial v}{\partial \theta}$ (b) $\frac{\partial u}{\partial \theta} = r\frac{\partial v}{\partial r}, \ \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$
(c) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ (d) $\frac{\partial u}{\partial r} = r\frac{\partial v}{\partial \theta}, \ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ **Ans.** (c)
(R.G.P.V., Bhopal, III Semester, June, 2007)

18. Analytic function is
(a) single valued function (b) bounded function
(c) differentiable function (d) All the above **Ans.** (d)

19. If $w = f(r e^{i\theta})$, then $\frac{dw}{dz}$ is
(a) $(\cos\theta + i\sin\theta)\frac{\partial w}{\partial r}$ (b) $(\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}$
(c) $(\sin\theta + i\cos\theta)\frac{\partial w}{\partial r}$ (d) $(\sin\theta - i\cos\theta)\frac{\partial w}{\partial r}$ **Ans.** (b)

20. If z_1 and z_2 are two complex numbers then $|z_1 + z_2|$ is
(a) $= |z_1| + |z_2|$ (b) $\leq |z_1| + |z_2|$
(c) $\leq |z_1| - |z_2|$ (d) $\geq |z_1| + |z_2|$ **Ans.** (b)

21. If $w = u(x, y) + i\,v(x, y)$ is an analytic function of $z = x + iy$, then $\frac{dw}{dz}$ equals
(a) $i\frac{\partial w}{\partial x}$ (b) $-i\frac{\partial w}{\partial x}$
(c) $i\frac{\partial w}{\partial y}$ (d) $-i\frac{\partial w}{\partial y}$ **Ans.** (d)

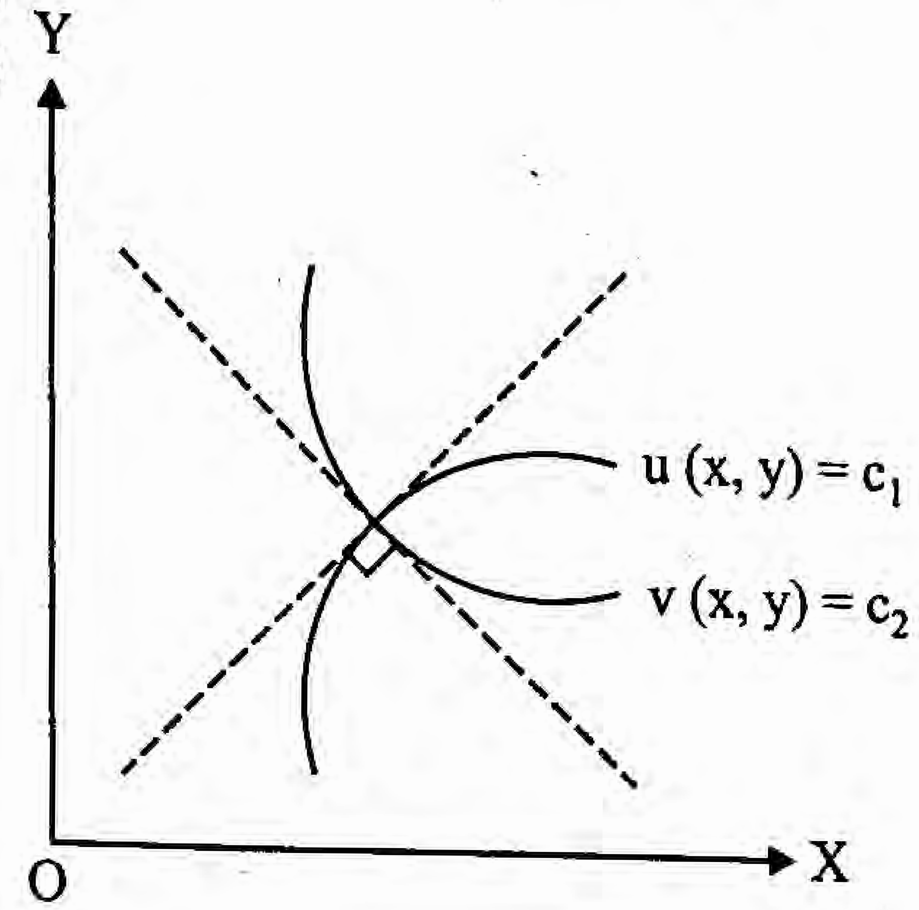
22. The curve $u(x, y) = C$ and $v(x, y) = C_1$ are orthogonal if
(a) u and v are complex functions (b) $u + iv$ is an analytic function.
(c) $u - v$ is an analytic function. (d) $u + v$ is an analytic function **Ans.** (b)
(U.P. II Semester, June 2009)

11.12 ORTHOGONAL CURVES

Two curves are said to be orthogonal to each other, when they intersect at right angle at each of their points of intersection.

The analytic function $f(z) = u + iv$ consists of two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ which form an orthogonal system.

$$u(x, y) = c_1 \quad \text{...(1)}$$
$$v(x, y) = c_2 \quad \text{... (2)}$$



Differentiating (1), $\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1 \text{ (say)}$$

Similarly from (2), $\frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = m_2$ *(say)*

The product of two slopes

$$m_1 m_2 = \left(-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)\left(-\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}\right) = \left(-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)\left(-\frac{-\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}\right) \quad (C-R \text{ equations})$$
$$= -1$$

Since $m_1 m_2 = -1$, two curves $u = c_1$ and $v = c_2$ are orthogonal, and c_1, c_2 are parameters, $u = c_1$ and $v = c_2$ form an orthogonal system.

11.13 HARMONIC FUNCTION *(U.P., II Semester 2009-2010)*

Any function which satisfies the Laplace's equation is known as a harmonic function.

Theorem. If $f(z) = u + iv$ is an analytic function, then u and v are both harmonic functions.

Proof. Let $f(z) = u + iv$, be an analytic function, then we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ...(1)$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad ...(2)$$
C – R equations.

Differentiating (1) with respect to x, we get $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ ... (3)

Differentiating (2) w.r.t. 'y' we have $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$ ... (4)

Adding (3) and (4) we have $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \left(\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}\right)$$

Similarly $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Therefore both u and v are harmonic functions.

Such functions u, v are called **Conjugate harmonic functions if $u + iv$ is also analytic function.**

Example 20. *Define a harmonic function and conjugate harmonic function. Find the harmonic conjugate function of the function $U(x, y) = 2x(1 - y)$.* (U.P., III Semester Dec. 2009)

Solution. See Art 11.13, on page 345. Here, we have $U(x, y) = 2x(1 - y)$.

Let V be the harmonic conjugate of U.

By total differentiation

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy \quad \text{(Total Differentiation)}$$
$$= -\frac{\partial U}{\partial y} dx + \frac{\partial U}{\partial x} dy$$
$$= -(-2x)\,dx + (2 - 2y)\,dy$$
$$= 2x\,dx + (2\,dy - 2y\,dy)$$
$$V = x^2 + 2y - y^2 + C$$

$$\left[U = 2x - 2xy,\quad \frac{\partial U}{\partial x} = 2 - 2y,\quad \frac{\partial U}{\partial y} = -2x\right]$$

Hence, the harmonic conjugate of U is $x^2 + 2y - y^2 + C$ **Ans.**

Example 21. *Show that the function $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic. Find its harmonic conjugate.*

Solution. $u = \frac{1}{2}\log(x^2 + y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}\cdot(2x) = \frac{x}{x^2+y^2} \quad \text{Similarly } \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2).1 - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2).1 - y(2y)}{(x^2+y^2)^2} = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence u is a harmonic function.

Let v be the harmonic conjugate of u.

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad \text{(By C – R equations)}$$
$$dv = -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$
$$dv = \frac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$$

Integrating, we get $v = \tan^{-1}\frac{y}{x} + C$, where C is a real constant.

This is the required harmonic conjugate. **Ans.**

Example 22. *Prove that $u = x^2 - y^2$ and $v = \frac{y}{x^2+y^2}$ are harmonic functions of (x, y), but are not harmonic conjugates.*

Solution. We have, $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$u(x, y)$ satisfies Laplace equation, hence $u(x, y)$ is harmonic

$$v = \frac{y}{x^2+y^2}, \quad \frac{\partial v}{\partial x} = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2(-2y) - (-2xy)2(x^2+y^2)2x}{(x^2+y^2)^4} \quad ... (1)$$
$$= \frac{(x^2+y^2)(-2y) - (-2xy)4x}{(x^2+y^2)^3} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)\cdot 1 - y(2y)}{(x^2+y^2)^2} = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2(-2y) - (x^2-y^2)2(x^2+y^2)(2y)}{(x^2+y^2)^4} = \frac{(x^2+y^2)(-2y) - (x^2-y^2)(4y)}{(x^2+y^2)^3}$$
$$= \frac{-2x^2y - 2y^3 - 4x^2y + 4y^3}{(x^2+y^2)^3} = \frac{-6x^2y + 2y^3}{(x^2+y^2)^3} \quad ... (2)$$

On adding (1) and (2), we get $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$v(x, y)$ also satisfies Laplace equations, hence $v(x, y)$ is also harmonic function.

But $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

Therefore u and v are not harmonic conjugates.

Proved.

Example 23. *Show that the function $x^2 - y^2 + 2y$ which is harmonic remains harmonic under the transformation $z = w^3$*

Solution.
$$u = x^2 - y^2 + 2y$$

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial^2 u}{\partial x^2} = 2 \qquad \ldots(1)$$

$$\frac{\partial u}{\partial y} = -2y + 2 \qquad \Rightarrow \qquad \frac{\partial^2 u}{\partial y^2} = -2 \qquad \ldots(2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence function is harmonic.

Transformation: $z = w^3$, $z = re^{i\theta}$ and $w = Re^{i\phi}$

$$\Rightarrow \quad re^{i\theta} = \left(Re^{i\phi}\right)^3 \quad \Rightarrow \quad re^{i\theta} = R^3 e^{3i\phi}$$

By comparing both side $r = R^3$, $\theta = 3\phi$

Given function, $f(x, y) = x^2 - y^2 + 2y$ where $x = r\cos\theta$ and $y = r\sin\theta$

$$f(r\cos\theta, r\sin\theta) = (r\cos\theta)^2 - (r\sin\theta)^2 + 2 \times r\sin\theta$$
$$= r^2\cos^2\theta - r^2\sin^2\theta + 2r\sin\theta$$
$$= r^2(\cos^2\theta - \sin^2\theta) + 2r\sin\theta = r^2\cos 2\theta + 2r\sin\theta$$

$$f(R^3\cos 3\phi, R^3\sin 3\phi) = R^6\cos 6\phi + 2R^3\sin 3\phi$$

This is a function in cosine and sine. Hence it will be harmonic function.

Proved.

Example 24. *If ϕ and ψ are functions of x and y satisfying Laplace's equation, show that $s + it$ is analytic, where*

$$s = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \quad \text{and} \quad t = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}.$$

Solution. Since ϕ and ψ are functions of x and y satisfying Laplace's equations,

$$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \ldots(1)$$

and
$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \qquad \ldots(2)$$

For the function $s + it$ to be analytic,

$$\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} \qquad \ldots(3)$$

and
$$\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x} \qquad \ldots(4)$$

must be satisfied.

Now,
$$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} \qquad \ldots(5)$$

$$\frac{\partial t}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2} \qquad \ldots(6)$$

$$\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} \qquad \ldots(7)$$

and
$$\frac{\partial t}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y} \qquad \ldots(8)$$

From (3), (5) and (6), we have

$$\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2} \quad \Rightarrow \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

which is true by (2).

Again from (4), (7) and (8), we have

$$\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} = -\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial y} \quad \Rightarrow \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

which is also true by (1).

Hence the function $s + it$ is analytic.

Proved.

Example 25. *If $u(x, y)$ and $v(x, y)$ are harmonic functions in a region R, prove that the function*

$$\left[\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]$$

is an analytic function of $z = x + iy$.

Solution. Since $u(x, y)$ and $v(x, y)$ are harmonic functions in a region R, therefore

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots(1)$$

and
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \ldots(2)$$

Let
$$F(z) = R + iS = \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Equating real and imaginary parts, we get

$$R = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}, \quad S = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

$$\frac{\partial R}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2} \qquad \ldots(3)$$

$$\frac{\partial R}{\partial y} = \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y} \qquad \ldots(4)$$

$$\frac{\partial S}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \qquad \ldots(5)$$

$$\frac{\partial S}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \qquad \ldots(6)$$

Putting the value of $\frac{\partial^2 u}{\partial x^2}$ from (1) in (5), we get

The magnitude of the resultant velocity

$$= \left|\frac{df}{dz}\right| = \sqrt{v_x^2 + v_y^2}.$$

$\phi(x, y) = C_1$ and $\psi(x, y) = C_2$ are called equipotential lines and lines of force respectively.

In heat flow problem the curves $\phi(x, y) = C_1$ and $\psi(x, y) = C_2$ are known as isotherms and heat flow lines respectively.

METHOD TO FIND THE CONJUGATE FUNCTION

Case I. Given. If $f(z) = u + iv$, and u is known.

To find. v, conjugate function.

Method. We know that $dv = \frac{\partial v}{\partial x}\cdot dx + \frac{\partial v}{\partial y}\cdot dy$ (Total Differentiation) ... (1)

Replacing $\frac{\partial v}{\partial x}$ by $-\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $\frac{\partial u}{\partial x}$ in (1), we get [C-R equations]

$$dv = -\frac{\partial u}{\partial y}\cdot dx + \frac{\partial u}{\partial x}\cdot dy$$

$$v = -\int \frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x}\cdot dy$$

$\Rightarrow$ $$v = \int M\, dx + \int N\, dy$$ ... (2)

where $$M = -\frac{\partial u}{\partial y} \text{ and } N = \frac{\partial u}{\partial x}$$

so that $$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$$

since u is a conjugate function, so $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\Rightarrow$ $$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ ... (3)

Equation (3) satisfies the condition of an exact differential equation.

So equation (2) can be integrated and thus v is determined.

Case II. Similarly, if $v = v(x, y)$ is given

To find out u.

We know that $$du = \frac{\partial u}{\partial x} dx + i\frac{\partial u}{\partial y} dy$$ ... (4)

On substituting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in (4), we get

$$du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$$

On integrating, we get

$$u = \int \frac{\partial v}{\partial y} dx - \int \frac{\partial v}{\partial x} dy$$ ...(5)

(since v is already known so $\frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}$ on R.H.S. are also known)

Equation (5) is an exact differential equation. On solving (5), u can be determined. Consequently $f(z) = u + iv$ can also be determined.

Example 26. *Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function. If $u = 3x - 2xy$, then find v and express $f(z)$ in terms of z.*

Solution. Here, we have $u = 3x - 2xy$

$$\frac{\partial u}{\partial x} = 3 - 2y, \qquad \frac{\partial u}{\partial y} = -2x$$

We know that $$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$ (Total differentiation)

$$= \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy \quad \left(\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}\right)$$

$$= 2x\, dx + (3 - 2y)\, dy$$

$$v = \int 2x\, dx + \int (3 - 2y)\, dy = x^2 + 3y - y^2 + c$$

$$f(z) = u(x, y) + iv(x, y) = (3x - 2xy) + i(x^2 + 3y - y^2 + c)$$

$$= (ix^2 - iy^2 - 2xy) + (3x + 3yi) + ic = i(x^2 - y^2 + 2ixy) + 3(x + iy) + ic$$

$$= i(x + iy)^2 + 3(x + iy) + ic = iz^2 + 3z + ic$$

Which is the required expression of $f(z)$ in terms of z. **Ans.**

Example 27. *Show that the function $u(x, y) = 4xy - 3x + 2$ is harmonic. Construct the corresponding analytic function $f(z) = u(x, y) + iv(x, y)$. Express $f(z)$ in terms of complex variable z.*

Solution. $u = 4xy - 3x + 2$ ... (1)

$$\frac{\partial u}{\partial x} = 4y - 3, \quad \frac{\partial^2 u}{\partial x^2} = 0,$$ ... (2)

$$\frac{\partial u}{\partial y} = 4x, \quad \frac{\partial^2 u}{\partial y^2} = 0$$ ... (3)

On adding (2) and (3), we get $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$u(x, y)$ satisfied Laplace equation, hence $u(x, y)$ is harmonic. **Proved.**

We know that $$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$ (Total differentiation)

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad \left[\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \text{ C-R equations}\right]$$

$$= -4x\, dx + (4y - 3)\, dy$$

$$v = \int -4x\, dx + \int (4y - 3)\, dy$$ (Exact differential equation)

$$= -2x^2 + 2y^2 - 3y + c$$

$f(z) = u(x, y) + iv(x, y)$

$= 4xy - 3x + 2 + i(-2x^2 + 2y^2 - 3y) + ic = -2ix^2 + 4xy + 2iy^2 - 3x - 3iy + 2 + ic$

$= -2i(x^2 + 2ixy - y^2) - 3(x + iy) + 2 + ic = -2i(x + iy)^2 - 3(x + iy) + 2 + ic$

$= -2iz^2 - 3z + 2 + ic$

Which is the required expression of $f(z)$ in terms of z. **Ans.**

Example 28. *Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function v such that $f(z) = u + iv$ is analytic. Also express $f(z)$ in terms of z.*

(R.G.P.V., Bhopal, III Semester, June 2005)

Solution. We have,

$$u = x^2 - y^2 - 2xy - 2x + 3y$$

$$\frac{\partial u}{\partial x} = 2x - 2y - 2 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y - 2x + 3 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since Laplace equation is satisfied, therefore u is harmonic. **Proved.**

We know that $\quad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$ [Total differentiation]

$$\Rightarrow \quad dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \qquad \ldots(1) \qquad \left[\because \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ and } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}\right]$$

Putting the values of $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x}$ in (1), we get [C-R equations]

$$dv = -(-2y - 2x + 3)\,dx + (2x - 2y - 2)dy$$

$$\Rightarrow \quad v = \int (2y + 2x - 3)dx + \int (-2y - 2)dy + C \qquad \text{(Ignoring } 2x\text{)}$$

Hence, $\quad v = 2xy + x^2 - 3x - y^2 - 2y + C$ **Ans.**

Now, $\quad f(z) = u + iv$

$$= (x^2 - y^2 - 2xy - 2x + 3y) + i(2xy + x^2 - 3x - y^2 - 2y) + iC$$

$$= (x^2 - y^2 + 2ixy) + (ix^2 - iy^2 - 2xy) - (2 + 3i)x - i(2 + 3i)y + iC$$

$$= (x^2 - y^2 + 2ixy) + i(x^2 - y^2 + 2ixy) - (2 + 3i)x - i(2 + 3i)y + iC$$

$$= (x + iy)^2 + i(x + iy)^2 - (2 + 3i)(x + iy) + iC$$

$$= z^2 + iz^2 - (2 + 3i)z + iC$$

$$= (1 + i)z^2 - (2 + 3i)z + iC$$

Which is the required expression of $f(z)$ in terms of z. **Ans.**

Example 29. *Define a harmonic function. Show that the function $u(x, y) = x^4 - 6x^2y^2 + y^4$ is harmonic. Also find the analytic function $f(z) = u(x, y) + iv(x, y)$.*

Solution. See Art. 11.13 on page 345 for definition of harmonic function.

We have,

$$u(x, y) = x^4 - 6x^2y^2 + y^4$$

$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2$$

$$\frac{\partial u}{\partial y} = -12x^2y + 4y^3$$

$$\frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2 \qquad \ldots(1)$$

$$\frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2 \qquad \ldots(2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12x^2 - 12y^2 - 12x^2 + 12y^2$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence, u is a harmonic function. **Proved.**

Let us find out v:

We know that $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$ (Total Differentiation)

$$\Rightarrow \quad dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \qquad \left[\because \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ and } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}\right]$$

$$\Rightarrow \quad dv = (12x^2y - 4y^3)\,dx + (4x^3 - 12xy^2)\,dy$$

$$v = \int (12x^2y - 4y^3)\,dx + \int (4x^3 - 12xy^2)\,dy$$

(y is constant) (Integrate only those terms which do not contain x)

$$v = 4x^3y - 4xy^3 + C$$

$$f(z) = u + iv$$

$$= x^4 - 6x^2y^2 + y^4 + i4x^3y - 4ixy^3 + iC$$

$$f(z) = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4 + iC$$

$$= (x + iy)^4 + iC$$

$$= z^4 + iC \qquad [\because z = x + iy]$$

This is the required analytic function **Ans.**

Example 30. *If $w = \phi + i\psi$ represents the complex potential for an electric field and*

$$\psi = x^2 - y^2 + \frac{x}{x^2 + y^2},$$

determine the function ϕ.

Solution. $\quad w = \phi + i\psi \quad$ and $\quad \psi = x^2 - y^2 + \frac{x}{x^2 + y^2}$

$$\frac{\partial \psi}{\partial x} = 2x + \frac{(x^2 + y^2)\cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial \psi}{\partial y} = -2y - \frac{x(2y)}{(x^2 + y^2)^2} = -2y - \frac{2xy}{(x^2 + y^2)^2}$$

We know that, $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = \frac{\partial \psi}{\partial y}dx - \frac{\partial \psi}{\partial x}dy$ (C – R equation)

$$= \left(-2y - \frac{2xy}{(x^2+y^2)^2}\right)dx - \left(2x + \frac{y^2-x^2}{(x^2+y^2)^2}\right)dy$$

$$\phi = \int\left[-2y - \frac{2xy}{(x^2+y^2)^2}\right]dx + c$$

This is an exact differential equation.

$$\phi = -2xy + \frac{y}{x^2+y^2} + C$$

Ans.

Which is the required function.

Example 31. *Construct the analytic function $f(z)$ of which the real part is $e^x \cos y$.*

Solution. $f(z) = u(x, y) + iv(x, y)$,

$$u(x, y) = e^x \cos y$$

$$\frac{\partial u}{\partial x} = e^x \cos y, \qquad \frac{\partial u}{\partial y} = -e^x \sin y$$

We know that, $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$ (Total differential)

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \qquad \left[\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \quad C-R \text{ equations}\right]$$

$$= e^x \sin y \, dx + e^x \cos y \, dy$$

This is an exact differential equation

$$v = \int e^x \sin y \cdot dx + \int e^x \cos y \, dy$$

[y as constant] [Ignoring the term containing x]

$$v = e^x \sin y$$

$$f(z) = u + iv = e^x \cos y + ie^x \sin y = e^x (\cos y + i \sin y)$$

$$= e^x \cdot e^{iy} = e^{x+iy} = e^z.$$

Ans.

Which is the required analytic function.

Example 32. *Find an analytic function $w = u + iv$ given that*

$$v = \frac{x}{x^2+y^2} + \cosh x \cos y.$$

Solution. $w = u + iv$

Given that $v = \frac{x}{x^2+y^2} + \cosh x \cos y$

We know that $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ (Total differentiation)

$$= \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \qquad \left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right)$$

On substituting the values of $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$, we have

$$du = \left(\frac{-2xy}{(x^2+y^2)^2} - \cosh x \sin y\right)dx - \left[\frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} + \sinh x \cos y\right]dy$$

$$= \left(\frac{-2xy}{(x^2+y^2)^2} - \cosh x \sin y\right)dx - \left[\frac{(y^2-x^2)}{(x^2+y^2)^2} + \sinh x \cos y\right]dy$$

This is an exact differential equation

$$\int du = \int\left(\frac{-2xy}{(x^2+y^2)^2} - \cosh x \sin y\right)dx - \int\left(\frac{y^2-x^2}{(x^2+y^2)^2} + \sinh x \cos y\right)dy$$

$$u = \frac{y}{x^2+y^2} - \sinh x \sin y + C$$

$$w = u + iv = \frac{y}{x^2+y^2} - \sinh x \sin y + i\left[\frac{x}{x^2+y^2} + \cosh x \cos y\right] + C$$

$$= \frac{y+ix}{x^2+y^2} - \sinh x \sin y + i \cosh x \cos y + C$$

$$= \frac{i(x-iy)}{x^2+y^2} + i \sin ix \sin y + i \cos ix \cos y + C$$

$$= \frac{i\bar{z}}{|z|^2} + i\cos(ix - y) + C = \frac{i\bar{z}}{|z|^2} + i \cos i(x+iy) + C$$

$$= \frac{i\bar{z}}{|z|^2} + i \cosh z + C$$

Ans.

Which is the required analytic function.

Example 33. *An electrostatic field in the xy-plane is given by the potential function $\phi = 3x^2y - y^3$, find the stream function.*

Solution. Let $\psi(x, y)$ be a stream function (*G.B.T.U, III Semester, 2012, U.P., Jan 2011*)

We know that $d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = \left(-\frac{\partial \phi}{\partial y}\right)dx + \left(\frac{\partial \phi}{\partial x}\right)dy$ [C-R equations]

$$= \{-(3x^2 - 3y^2)\}\,dx + 6xy\,dy$$

$$= -3x^2\,dx + (3y^2\,dx + 6xy\,dy)$$

$$= -d(x^3) + 3d(xy^2)$$

$$\psi = \int -d(x^3) + 3d(xy^2) + c$$

$$\psi = -x^3 + 3xy^2 + c$$

Ans.

ψ is the required stream function.

Example 34. *In a two-dimensional fluid flow, the stream function is $\psi = -\frac{y}{x^2+y^2}$, find the velocity potential ϕ.*

Solution. $\psi = -\frac{y}{x^2+y^2}$ ...(1)

$$\frac{\partial \psi}{\partial x} = \frac{2xy}{(x^2+y^2)^2}, \qquad \frac{\partial \psi}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

We know that,
$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = \frac{\partial \psi}{\partial y}dx - \frac{\partial \psi}{\partial x}dy \qquad \text{[C-R equations]}$$
$$= \frac{(y^2-x^2)}{(x^2+y^2)^2}dx - \frac{2xy}{(x^2+y^2)^2}dy$$
$$= \frac{(x^2+y^2)\,dx - 2x^2\,dx - 2xy\,dy}{(x^2+y^2)^2}$$
$$= \frac{(x^2+y^2)dx - x\,(2x\,dx + 2y\,dy)}{(x^2+y^2)^2}$$
$$= \frac{(x^2+y^2)\,d(x) - x\,d\,(x^2+y^2)}{(x^2+y^2)^2} = d\left(\frac{x}{x^2+y^2}\right)$$
$$\Rightarrow \qquad \phi = \int d\left(\frac{x}{x^2+y^2}\right)$$
$$\Rightarrow \qquad \phi = \frac{x}{x^2+y^2} + c$$

ϕ is the required velocity potential. **Ans.**

Example 35. *Find the imaginary part of the analytic function whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2$.*

(*R.G.P.V., Bhopal, III Semester, June 2009, Dec. 2008, 2005*)

Solution. Let $u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$
$$\frac{\partial u}{\partial y} = -6xy - 6y$$

We know that
$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \quad \Rightarrow \quad dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \qquad \text{[Total differentiation]}$$
$$\Rightarrow \qquad dv = (6xy + 6y)\,dx + (3x^2 - 3y^2 + 6x)\,dy$$

This is an exact differential equation.
$$v = \int (6xy + 6y)\,dx + \int -3y^2 dy + C$$
$$= 3x^2y + 6xy - y^3 + C$$

Which is the required imaginary part. **Ans.**

Example 36. *If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z.*

Solution. $u + iv = f(z) \quad \Rightarrow \quad iu - v = if(z)$

Adding these, $(u - v) + i(u + v) = (1 + i)f(z)$

Let
$$U + iV = (1+i)f(z) \quad \text{where } U = u - v \text{ and } V = u + v$$
$$F(z) = (1+i)f(z)$$
$$U = u - v = (x - y)(x^2 + 4xy + y^2)$$
$$= x^3 + 3x^2y - 3xy^2 - y^3$$

$$\frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2$$
$$\frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2$$

We know that
$$dV = \frac{\partial V}{\partial x}\cdot dx + \frac{\partial V}{\partial y}dy = -\frac{\partial U}{\partial y}\cdot dx + \frac{\partial U}{\partial x}\cdot dy \qquad \text{[C-R equations]}$$

On putting the values of $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$, we get
$$= (-3x^2 + 6xy + 3y^2)\,dx + (3x^2 + 6xy - 3y^2)\,.\,dy$$

Integrating, we get
$$V = \int(-3x^2 + 6xy + 3y^2)dx + \int(-3y^2)dy$$
(y as constant) (Ignoring terms of x)
$$= -x^3 + 3x^2y + 3xy^2 - y^3 + c$$
$$F(z) = U + iV$$
$$= (x^3 + 3x^2y - 3xy^2 - y^3) + i(-x^3 + 3x^2y + 3xy^2 - y^3) + ic$$
$$= (1-i)x^3 + (1+i)\,3x^2y - (1-i)\,3xy^2 - (1+i)y^3 + ic$$
$$= (1-i)x^3 + i(1-i)\,3x^2y - (1-i)\,3xy^2 - i(1-i)y^3 + ic$$
$$= (1-i)[x^3 + 3ix^2y - 3xy^2 - iy^3] + ic$$
$$= (1-i)(x+iy)^3 + iC = (1-i)z^3 + ic$$
$$(1+i)f(z) = (1-i)z^3 + ic, \qquad [F(z) = (1+i)f(z)]$$
$$f(z) = \frac{1-i}{1+i}z^3 + \frac{ic}{1+i} = -\frac{i(1+i)}{(1+i)}z^3 + \frac{i(1-i)}{(1+i)(1-i)}c = -iz^3 + \frac{1+i}{2}c \qquad \textbf{Ans.}$$

Example 37. *If $f(z) = u + iv$, is any analytic function of the complex variable z and $u - v = e^x(\cos y - \sin y)$, find $f(z)$ in terms of z.*

Solution. $u + iv = f(z) \quad \Rightarrow \quad iu - v = if(z)$

Adding, we have
$$u + iv + iu - v = f(z) + if(z)$$
$$(u - v) + i(u + v) = (1 + i)f(z) = F(z) \text{ say}$$

Put $u - v = U$ and $u + v = V$, then $F(z) = U + iV$ is an analytic function.

Now $U = e^x(\cos y - \sin y)$

$$\therefore \qquad \frac{\partial U}{\partial x} = e^x(\cos y - \sin y) \quad \text{and} \quad \frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$$

We know that
$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy = -\frac{\partial U}{\partial y}dx + \frac{\partial U}{\partial x}dy$$
$$= e^x(\sin y + \cos y)\,dx + e^x(\cos y - \sin y)dy.$$

Integrating, we have
$$V = e^x(\sin y + \cos y) + c$$
$$F(z) = U + iV$$
$$= e^x(\cos y - \sin y) + ie^x(\sin y + \cos y) + ic$$

$$\frac{\partial u}{\partial r} = -2r\sin 2\theta + \sin\theta$$

⇒ $$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta \quad \ldots(4)$$

[From (3)]

⇒ $$\frac{\partial u}{\partial \theta} = -2r^2\cos 2\theta + r\cos\theta \quad \ldots(5)$$

By total differentiation formula

$$du = \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta = (-2r\sin 2\theta + \sin\theta)dr + (-2r^2\cos 2\theta + r\cos\theta)d\theta$$

$$= -[(2r\,dr)\sin 2\theta + r^2(2\cos 2\theta\, d\theta)] + [\sin\theta \cdot dr + r(\cos\theta\, d\theta)]$$

$$= -[(2r\,dr)\sin 2\theta - \sin\theta dr] + [-r^2 2\cos 2\theta\, d\theta + r\cos\theta\, d\theta]$$

$$= -d(r^2\sin 2\theta) + d(r\sin\theta) \quad \text{(Exact differential equation)}$$

Integrating, we get

$$u = -r^2\sin 2\theta + r\sin\theta + c$$

Hence, $$f(z) = u + iv = (-r^2\sin 2\theta + r\sin\theta + c) + i(r^2\cos 2\theta - r\cos\theta + 2)$$

$$= ir^2(\cos 2\theta + i\sin 2\theta) - ir(\cos\theta + i\sin\theta) + 2i + c$$

$$= ir^2e^{2i\theta} - ir\,e^{i\theta} + 2i + c = i(r^2e^{2i\theta} - r\,e^{i\theta}) + 2i + c.$$

This is the required analytic function. **Ans.**

Example 41. *Deduce the following with the polar form of Cauchy-Riemann equations :*

(a) $$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$ (b) $$f'(z) = \frac{r}{z}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$

Solution. We know that polar form of C-R equations is

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \ldots(1)$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \ldots(2)$$

(a) Differentiating (1) partially w.r.t. r, we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r\partial \theta} \quad \ldots(3)$$

Differentiating (2) partially w.r.t. θ, we have

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial \theta\partial r} \quad \ldots(4)$$

Thus, using (1), (3) and (4) we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r\partial \theta} + \frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) + \frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial \theta\partial r}\right) = 0 \quad \left[\frac{\partial^2 v}{\partial \theta\partial r} = \frac{\partial^2 v}{\partial \theta\partial r}\right]$$

Proved.

(b) Now, $$r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = r\left[\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right)\right]$$

$$= r\left[\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + i\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\right]$$

$$= r\cos\theta\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + r\sin\theta\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \quad \text{[By C – R equation]}$$

$$= x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + iy\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)$$

$$= x f'(z) + iy f'(z) = (x + iy) f'(z) = z f'(z).$$

∴ $$f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Proved.

11.18 MILNE THOMSON METHOD (To construct an Analytic function)

By this method $f(z)$ is directly constructed without finding v and the method is given below:

Since $z = x + iy$ and $\bar{z} = x - iy$

∴ $$x = \frac{z + \bar{z}}{2},\ y = \frac{z - \bar{z}}{2i}$$

$$f(z) \equiv u(x, y) + iv(x, y)$$

$$f(z) \equiv u\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) + iv\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) \quad \ldots(1)$$

This relation can be regarded as a formal identity in two independent variables z and $\bar{z}$. Replacing $\bar{z}$ by z, we get

$$f(z) = u(z, 0) + iv(z, 0)$$

Which can be obtained by replacing x by z and y by 0 in (1)

Case I. If *u* is given

We have $$f(z) = u + iv$$

∴ $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad (C - R \text{ equations})$$

If we write $$\frac{\partial u}{\partial x} = \phi_1(x, y), \quad \frac{\partial u}{\partial y} = \phi_2(x, y)$$

$$f'(z) = \phi_1(x, y) - i\phi_2(x, y) \quad \text{or} \quad f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

On integrating $$f(z) = \int \phi_1(z, 0)\, dz - i\int \phi_2(z, 0)dz + c$$

Case II. If *v* is given

$$f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \psi_1(x, y) + i\,\psi_2(x, y)$$

when $$\psi_1(x, y) = \frac{\partial v}{\partial y}, \quad \psi_2(x, y) = \frac{\partial v}{\partial x}.$$

$$f(z) = \int \psi_1(z, 0)\, dz + i\int \psi_2(z, 0)\, dz + c$$

WORKING RULE: TO CONSTRUCT AN ANALYTIC FUNCTION BY MILNE THOMSON METHOD

Case I. When u is given

Step 1. Find $\frac{\partial u}{\partial x}$ and equate it to $\phi_1(x, y)$.

Step 2. Find $\frac{\partial u}{\partial y}$ and equate it to $\phi_2(x, y)$.

Step 3. Replace x by z and y by 0 in $\phi_1(x, y)$ to get $\phi_1(z, 0)$.

Step 4. Replace x by z and y by 0 in $\phi_2(x, y)$ to get $\phi_2(z, 0)$.

Step 5. Find $f(z)$ by the formula $f(z) = \int \{\phi_1(z, 0) - i\phi_2(z, 0)\}\, dz + c$

Case II. When v is given

Step 1. Find $\frac{\partial v}{\partial x}$ and equate it to $\psi_1(x, y)$.

Step 2. Find $\frac{\partial v}{\partial y}$ and equate it to $\psi_2(x, y)$.

Step 3. Replace x by z and y by 0 in $\psi_1(x, y)$ to get $\psi_1(z, 0)$.

Step 4. Replace x by z and y by 0 in $\psi_2(x, y)$ to get $\psi_2(z, 0)$.

Step 5. Find $f(z)$ by the formula

$$f(z) = \int \{\psi_1(z, 0) + i\psi_2(z, 0)\}\, dz + c$$

Case III. When $u - v$ is given.

We know that
$$f(z) = u + iv \quad ...(1)$$
$$if(z) = iu - v \quad ...(2) \text{ [Multiplying by } i]$$

Adding (1) and (2), we get

$$(1 + i)\, f(z) = (u - v) + i(u + v)$$

$$\Rightarrow \quad F(z) = U + iV$$

where
$$F(z) = (1 + i)\, f(z) \quad ...(3) \quad \begin{bmatrix} U = u - v \\ V = u + v \end{bmatrix}$$

Here, $U = (u - v)$ is given

Find out F(z) by the method described in case I, then substitute the value of F(z) in (3), we get

$$f(z) = \frac{F(z)}{1+i}$$

Case IV. When $u + v$ is given.

We know that
$$f(z) = u + iv \quad ...(1)$$
$$if(z) = iu - v \quad \text{[Multiplying by } i]...(2)$$

Adding (1) and (2), we get

$$(1 + i)\, f(z) = (u - v) + i(u + v)$$

$$\Rightarrow \quad F(z) = U + iV$$

where
$$F(z) = (1 + i)\, f(z) \quad ...(3) \quad \begin{bmatrix} U = u - v \\ V = u + v \end{bmatrix}$$

Here, $V = (u + v)$ is given

Find out F(z) by the method described in case II, then substitute the value of $f(z)$ in (3), we get

$$f(z) = \frac{F(z)}{1+i}$$

Example 42. *If $u = x^2 - y^2$, find a corresponding analytic function.*

Solution.
$$\frac{\partial u}{\partial x} = 2x = \phi_1(x, y), \qquad \frac{\partial u}{\partial y} = -2y = \phi_2(x, y)$$

On replacing x by z and y by 0, we have

$$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]dz$$
$$= \int [2z - i(0)]dz = \int 2z\, dz = z^2 + C$$

This is the required analytic function. **Ans.**

Example 43. *Show that the function $u = e^{-2xy} \sin(x^2 - y^2)$ is harmonic. Find the conjugate function v and express $u + iv$ as an analytic function of z.*

(R.G.P.V. Bhopal, III Semester, June, 2012, 2007, Dec. 2006)

Solution. We have,
$$u = e^{-2xy} \sin(x^2 - y^2) \quad ... (1)$$

Differentiating (1), w.r.t. x, we get

$$\frac{\partial u}{\partial x} = 2x\, e^{-2xy} \cos(x^2 - y^2) - 2y e^{-2xy} \sin(x^2 - y^2)$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = e^{-2xy} [2x \cos(x^2 - y^2) - 2y \sin(x^2 - y^2)] = \phi_1(x, y) \quad ... (2)$$

$$\phi_1(z, 0) = 2z \cos z^2$$

Differentiating (1), w.r.t. y, we get

$$\frac{\partial u}{\partial y} = -2y e^{-2xy} \cos(x^2 - y^2) - 2x e^{-2xy} \sin(x^2 - y^2)$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = e^{-2xy} [-2y \cos(x^2 - y^2) - 2x \sin(x^2 - y^2)] = \phi_2(x, y) \quad ... (3)$$

$$\phi_2(z, 0) = -2z \sin z^2$$

Differentiating (2), w.r.t, 'x', we get

$$\frac{\partial^2 u}{\partial x^2} = -2y e^{-2xy} [2x \cos(x^2 - y^2) - 2y \sin(x^2 - y^2)]$$
$$+ e^{-2xy} [2 \cos(x^2 - y^2) + 2x\,(2x)\, \{-\sin(x^2 - y^2)\} - 2y\,(2x) \cos(x^2 - y^2)]$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = e^{-2xy} [-4xy \cos(x^2 - y^2) + 4y^2 \sin(x^2 - y^2) + 2 \cos(x^2 - y^2)$$
$$- 4x^2 \sin(x^2 - y^2) - 4xy \cos(x^2 - y^2)]$$

$$= e^{-2xy} [-8xy \cos(x^2 - y^2) + 4y^2 \sin(x^2 - y^2) + 2 \cos(x^2 - y^2) - 4x^2 \sin(x^2 - y^2)] \quad ...(4)$$

Differentiating (3), w.r.t. 'y', we get

$$\frac{\partial^2 u}{\partial y^2} = -2x e^{-2xy} [-2y \cos(x^2 - y^2) - 2x \sin(x^2 - y^2)]$$
$$+ e^{-2xy} \left[-2 \cos(x^2 - y^2) + 2y\,(-2y) \sin(x^2 - y^2) - 2x(-2y) \cos(x^2 - y^2)\right]$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = e^{-2xy} [4xy \cos(x^2 - y^2) + 4x^2 \sin(x^2 - y^2) - 2 \cos(x^2 - y^2)$$
$$- 4y^2 \sin(x^2 - y^2) + 4xy \cos(x^2 - y^2)]$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-2xy} [8xy \cos(x^2 - y^2) + 4x^2 \sin(x^2 - y^2) - 2 \cos(x^2 - y^2) - 4y^2 \sin(x^2 - y^2)] \quad ... (5)$$

Adding (4) and (5), we get $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Which proves that u is harmonic.

Now we have to express $u + iv$ as a function of z

$$f(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\,dz$$

$$= \int [2z\cos z^2 - i(-2z\sin z^2)]\,dz$$

$$= \sin z^2 - i\cos z^2 + C$$

$$= -i(\cos z^2 + i\sin z^2) + C$$

$$= -i e^{iz^2} + C$$

Ans.

Example 44. *If* $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$, *find* $f(z)$.

Solution. $\dfrac{\partial u}{\partial x} = \dfrac{(\cosh 2y + \cos 2x)2\cos 2x - \sin 2x(-2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$

$$= \frac{2\cosh 2y\cos 2x + 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y + \cos 2x)^2} = \frac{2\cosh 2y\cos 2x + 2}{(\cosh 2y + \cos 2x)^2} = \phi_1(x, y) \quad \dots(1)$$

Now putting $x = z$, $y = 0$ in (1), we get

$$\phi_1(z, 0) = \frac{2\cos 2z + 2}{(1 + \cos 2z)^2}$$

Again $\dfrac{\partial u}{\partial y} = \dfrac{-\sin 2x(2\sinh 2y)}{(\cosh 2y + \cos 2x)^2} = \dfrac{-2\sin 2x\sinh 2y}{(\cosh 2y + \cos 2x)^2} = \phi_2(x, y) \quad \dots(2)$

Now putting $x = z$, $y = 0$ in (2), we get

$$\phi_2(z, 0) = 0$$

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz = \int \frac{(2\cos 2z + 2)}{(1 + \cos 2z)^2}\,dz = 2\int \frac{1}{1 + \cos 2z}\,dz$$

$$= 2\int \frac{1}{2\cos^2 z}\,dz = \int \sec^2 z\,dz = \tan z + C$$

which is the required function. **Ans.**

Example 45. *Find the analytic function* $f(z) = u + iv$, *given that* $v = e^x(x\sin y + y\cos y)$.

Solution. $\dfrac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + e^x\sin y = \psi_2(x, y) \Rightarrow \psi_2(x, y) = \psi_2(z, 0) = 0$

$\dfrac{\partial v}{\partial y} = e^x(x\cos y + \cos y - y\sin y) = \psi_1(x, y) \Rightarrow \psi_1(x, y) = \psi_1(z, 0) = ze^z + e^z$

$$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]\,dz$$

$$= \int [e^z(z+1) + i(0)]\,dz = (z+1)e^z - \int e^z\,dz$$

$$= (z+1)e^z - e^z + C = z e^z + C$$

Which is the required function. **Ans.**

Example 46. *Show that* $e^x(x\cos y - y\sin y)$ *is a harmonic function. Find the analytic function for which* $e^x(x\cos y - y\sin y)$ *is imaginary part.*

(U.P., III Semester, June 2009, R.G.P.V., Bhopal, III Semester, Feb. 2010)

Solution. Here $v = e^x(x\cos y - y\sin y)$

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial v}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y = \psi_2(x, y), \quad \text{(say)} \quad \dots(1)$$

$$\frac{\partial v}{\partial y} = e^x(-x\sin y - y\cos y - \sin y) = \psi_1(x, y) \quad \text{(say)} \quad \dots(2)$$

$$\frac{\partial^2 v}{\partial x^2} = e^x(x\cos y - y\sin y) + e^x\cos y + e^x\cos y$$

$$= e^x(x\cos y - y\sin y + 2\cos y) \quad \dots(3)$$

and $\dfrac{\partial^2 v}{\partial y^2} = e^x(-x\cos y + y\sin y - 2\cos y) \quad \dots(4)$

Adding equations (3) and (4), we have

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \Rightarrow v \text{ is a harmonic function.}$$

Now putting $x = z$, $y = 0$ in (1) and (2), we get

$$\psi_2(z, 0) = ze^z + e^z, \qquad \psi_1(z, 0) = 0$$

Hence by Milne-Thomson method, we have

$$f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)]\,dz + C$$

$$= \int [0 + i(ze^z + e^z)]\,dz + C = i(ze^z - e^z + e^z) + C = ize^z + C.$$

This is the required analytic function. **Ans.**

Example 47. *If* $u - v = (x - y)(x^2 + 4xy + y^2)$ *and* $f(z) = u + iv$ *is an analytic function of* $z = x + iy$, *find* $f(z)$ *in terms of* z *by Milne Thomson method.*

Solution. We know that

$$f(z) = u + iv \quad \dots(1)$$

$$i f(z) = iu - v \quad \dots(2)$$

Adding (1) and (2), we get

$$(1 + i) f(z) = (u - v) + i(u + v)$$

$$F(z) = U + iV$$

$$\left[\begin{array}{r} (1+i)\,f(z) = F(Z) \\ u - v = U \\ u + v = V \end{array}\right]$$

$$U = u - v = (x - y)(x^2 + 4xy + y^2)$$

$$\frac{\partial U}{\partial x} = (x^2 + 4xy + y^2) + (x - y)(2x + 4y)$$

$$= x^2 + 4xy + y^2 + 2x^2 + 4xy - 2xy - 4y^2$$

$$= 3x^2 + 6xy - 3y^2$$

$$\phi_1(x, y) = 3x^2 + 6xy - 3y^2$$

$$\phi_1(z, 0) = 3z^2$$

$$\frac{\partial U}{\partial y} = -(x^2 + 4xy + y^2) + (x - y)(4x + 2y)$$

$$= -x^2 - 4xy - y^2 + 4x^2 + 2xy - 4xy - 2y^2$$

$$= 3x^2 - 6xy - 3y^2$$

$$\phi_2(x, y) = 3x^2 - 6xy - 3y^2$$

$$\phi_2(z, 0) = 3z^2$$

$$F(z) = U + iV$$

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$$

Multiplying by i, we get

$$if(z) = iu - v \quad \ldots(2)$$

Adding (1) and (2), we get

$$(1 + i) f(z) = (u - v) + i(u + v) \quad \ldots(3) \quad \begin{bmatrix} U = u - v \\ V = u + v \end{bmatrix}$$

$$F(z) = U + iV$$

$$\Rightarrow \quad F(z) = (1 + i) f(z) \quad \ldots(4)$$

where

$$u + v = V = \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x} \quad \ldots(5)$$

$$\Rightarrow \quad V = \frac{2\sin 2x}{2\cosh 2y - 2\cos 2x} \quad \ldots(6)$$

$$\Rightarrow \quad V = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

Here, V is given and we have to find out F(z) by Milne Thomson method

Now,

$$\frac{\partial V}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} = \frac{\partial U}{\partial x} = \psi_1(x, y) \quad \text{(say)}$$

and

$$\frac{\partial V}{\partial x} = \frac{2\cos 2x(\cosh 2y - \cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} = \psi_2(x, y) \quad \text{(say)}$$

On replacing x by z and y by 0 in $\psi_1(x, y)$, we get

$$\therefore \quad \psi_1(z, 0) = 0$$

On replacing x by z and y by 0 in $\psi_2(x, y)$, we get

$$\psi_2(z, 0) = \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2} = \frac{-2}{1 - \cos 2z} = \frac{-2}{1 - 1 + 2\sin^2 z} = -\operatorname{cosec}^2 z$$

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$$

By Milne Thomson method, we have

$$F(z) = \int \{\psi_1(z, 0) + i\,\psi_2(z, 0)\}\, dz$$

$$= \int -i \operatorname{cosec}^2 z\, dz = i \cot z + c$$

Replacing F(z) by $(1 + i) f(z)$, from equation (5), we get

$$(1 + i) f(z) = i \cot z + c$$

$$\Rightarrow \quad f(z) = \frac{i}{1+i}\cot z + \frac{c}{1+i}$$

$$\therefore \quad f(z) = \frac{1}{2}(1+i)\cot z + c_1 \quad \text{where } c_1 = \frac{c}{1+i} \quad \textbf{Ans.}$$

which is the required function.

EXERCISE 11.2

Show that the following functions are harmonic and determine the conjugate functions.

1. $u = 2x(1 - y)$ **Ans.** $v = x^2 - y^2 + 2y + C$
2. $u = 2x - x^3 + 3xy + 3xy^2$ **Ans.** $v = -\frac{3}{2}(x^2 - y^2) + \frac{3}{2}y^2 + y^3 + 2y + C$

Determine the analytic function, whose real part is

3. $2x(1 - y)$ **Ans.** $i z^2 + 2z + C$
4. $\log\sqrt{x^2 + y^2}$ (K.U., 2009) **Ans.** $\log z + C$
5. $x^2 - y^2 + 5x + y - \frac{y}{x^2 + y^2}$ **Ans.** $z^2 + (5 - i)z - \frac{i}{z} + C$
6. $\cos x \cosh y$ **Ans.** $\cos z + c$
7. $3x^2 y + 2x^2 - y^3 - 2y^2$ **Ans.** $2z^2 - iz^3 + C$
8. $x^3 - 3xy^2 + 3x^2 - 3y^2 + 2x + 1$ **Ans.** $z^3 + 3z^2 + 2z + C$
9. $e^{-x}(x\sin y - y\cos y)$ **Ans.** $i(ze^{-z} + C)$
10. $e^{2x}(x\cos 2y - y\sin 2y)$ **Ans.** $z e^{2z} + iC$
11. $e^{-x}(x\cos y + y\sin y)$ and $f(0) = i$. **Ans.** $ze^{-z} + i$

Determine the analytic function, whose imaginary part is

12. $v = \log(x^2 + y^2) + x - 2y$ (G.B.T.U., 2012) **Ans.** $2i\log z - (2 - i)z + C$
13. $v = \sinh x \cos y$ **Ans.** $\sin iz + C$
14. $v = \frac{x - y}{x^2 + y^2}$ **Ans.** $(1 + i)\frac{1}{z} + C$
15. $v = -\frac{y}{x^2 + y^2}$ **Ans.** $\frac{1}{z} + C$
16. $v = \left(r - \frac{1}{r}\right)\sin\theta$ **Ans.** $z + \frac{1}{z} + C$
17. If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find $f(z)$ subject to the condition that $f\left(\frac{\pi}{2}\right) = \frac{3 - i}{2}$. **Ans.** $f(z) = \cot\frac{z}{2} + \frac{1 - i}{2}$
18. Find an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ such that $V(r, \theta) = r^2\cos 2\theta - r\cos\theta + 2$. **Ans.** $i\,[z^2 - z + 2]$
19. Show that the function $u = x^2 - y^2 - 2xy - 2x - y - 1$ is harmonic. Find the conjugate harmonic function v and express $u + iv$ as a function of z where $z = x + iy$. **Ans.** $v = x^2 - y^2 + 2xy + x - 2y$; $(1 + i)z^2 + (-2 + i)z - 1$
20. Construct an analytic function of the form $f(z) = u + iv$, where v is $\tan^{-1}(y/x)$, $x \neq 0, y \neq 0$. **Ans.** $\log cz$
21. Show that the function $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic. Find the conjugate function v and express $u + iv$ as an analytic function of z. **Ans.** $v = e^{-2xy}\cos(x^2 - y^2) + C$; $f(z) = -ie^{iz^2} + C_1$